Order of the complex numbers and its consequences

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From the formula $e^{x \cdot i} = \cos x + i \bullet \sin x$ Euler developed the power of the complex numbers. Nowadays, in the self construction of basic Mathematics (Mathematical logic, Set theory, Relations and structures and the Numerical system ["Mathematical Atlas" by Reinhard and Soeder, Publisher Alianza]) a didactic lack is presented because the formula from Euler can not be included in its development to build the power of the complex numbers.

In the essay that is resumed here, this didactic lack is avoided by means of an order of the complex numbers (modular order) from which, by means of sequences of powers of complex number base and rational number index of denominator power of two, the power of complex numbers with real numbers as index is defined. And by means of an inverse process to the one followed by Euler, the trigonometry is very easily developed. And finally, with the support of thetrigonometry already built, the pi number can be defined.

1.Quadrants

Being C the set of the complex numbers, C' the subset of the complex numbers of modulus one and R^+ the set of the

positive real numbers; if $k \in R^+$,we call $k \bullet C'$ the subset of the elements of C of modulus k.

And over $k \bullet C'$ we define the following subsets:

$$k \bullet C'_{-2} = \{x_1 + x_2 \bullet i \mid x_1 \in [-k, 0 [\wedge x_2 \in]-k, 0] \}$$
$$k \bullet C'_{-1} = \{x_1 + x_2 \bullet i \mid x_1 \in [0, k[\wedge x_2 \in [-k, 0 [\}$$
$$k \bullet C'_{0} = \{x_1 + x_2 \bullet i \mid x_1 \in] 0, k] \wedge x_2 \in [0, k [\}$$
$$k \bullet C'_{1} = \{x_1 + x_2 \bullet i \mid x_1 \in]-k, 0] \wedge x_2 \in]0, k] \}$$

Which we call quadrants of $k \bullet C'$ (If $k = 1,\ k \bullet C' = C'$).

Obviously C'_{-2}, C'_{-1}, C'_0 and C'_1 would be the quadrants of C'.

From now on, we agree: If $a \in C$, then $a' \in C'$, $a'' \in C'_0$ and $a = a_1 + a_2 \bullet i$.

1.1 Partition in $k \bullet C'$

The family of quadrants $\{k \bullet C'_{-2},\ k \bullet C'_{-1},\ k \bullet C'_0,\ k \bullet C'_1\}$ form a partition in $k \bullet C'$.

1.2. Canonical form of the complex numbers of modulus one.

Any $a' \in C'$ can be uniquely written in the form

$$a' = i^{c(a')} \bullet a''$$

in wich $c(a') \in \{-2, -1, 0, 1\}$ and $a'' \in C'_0$ $(a''_1 \in]0, 1]$ and $a''_2 \in [0, 1[)$.

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1.2.1 Canonical form of any complex number different from zero.

In which $c(a') \in \{-2,-1,0,1\}$ and $a'' \in C'_0$ $(a''_1 \in]0,1]$ and $a''_2 \in [0,1[)$.

1.2.1 Canonical form of any complex number different from zero.

Any $a \in C - \{0\}$ can be uniquely written in the form

$a = \|a\| \bullet i^{c(a)} \bullet a''$ $(a = \|a\| \bullet a')$

In which $\|a\|$ is the modulus of a, $c(a) \in \{-2,-1,0,1\}$ and $a'' \in C'_0$ $(c(a) = c(a'))$.

1.2.2 a) $c(a') = -2 \Leftrightarrow a' \in C'_{-2}$ b) $c(a') = -1 \Leftrightarrow a' \in C'_{-1}$

c) $c(a') = 0 \Leftrightarrow a' \in C'_0$ d) $c(a') = 1 \Leftrightarrow a' \in C'_1$

2. Modular order in C' and in C

1º Modular order in C'

Given the interval $[-2,2[$ in R (real number set),the application

$f : [-2,2[\rightarrow C'$ such that

$x \in [-2,2[\Rightarrow f(x) = i^{c(x)} \bullet (m'(x) + m(x) \bullet i)$ is bijective,

being $x = c(x) + m(x)$ in which $c(x)$ is the characteristic of the real number x and $m(x)$ the mantissa, and therefore $c(x) \in \{-2,-1,0,1\}$ and $m(x) \in [0,1[$ .The real number $m'(x)$,which

we will call the complementary of the mantissa $m(x)$, is $m'(x) = \sqrt{1 - m(x)^2}$

$(m'(x) + m(x) \bullet i \in C'_0)$.

This bijection allows us to define the order $<'$ in C' inferring isomorphically from the order $<$ in $[-2,2[$ the order $<'$ in C',in such a way that,for $x, y \in [-2,2[$

$x < y \Leftrightarrow i^{c(x)} \bullet (m'(x) + m(x) \bullet i) <' i^{c(y)} \bullet (m'(y) + m(y) \bullet i)$

Obviously

$i^{c(a')} \bullet (a''_1 + a''_2 \bullet i) <' i^{c(b')} \bullet (b''_1 + b''_2 \bullet i) \Leftrightarrow c(a') + a''_2 < c(b') + b''_2$

2º Modular order in C of the following way:

If $a, b \in C$ with $a = \|a\| \bullet a'$ and $b = \|b\| \bullet b'$

$a <' b \Leftrightarrow \|a\| < \|b\| \vee (\|a\| = \|b\| \wedge a' <' b')$

That is the lexicographic order inferred from the order $<$ in R and from the order $<'$ in C'.

If $\|a\| \bullet i^{c(a)} \bullet (a''_1 + a''_2 \bullet i)$ is the canonical representation of the complex number a ,we also give

to the integer number $c(a) \in \{-2,-1,0,1\}$ and to the real number $a''_2 \in [0,1[$, as seen before,

the respective names of characteristic and mantissa of that complex number a .We also agree

that $m(a) = a''_2$ and $m'(a) = a''_1$.

2.2.1 $a' <' b' \Leftrightarrow c(a') < c(b') \vee (c(a') = c(b') \wedge a''_2 < b''_2)$

2.2 a) $a' <' 1 \Leftrightarrow a' \in C'_{-2} \cup C'_{-1}$ b) $1 \leq' a' \Leftrightarrow a' \in C'_0 \cup C'_1$

2.3 The order $<'$ is total

$\forall_{x,y \in C}(x \leq' y \vee y \leq' x) \quad (x \leq' y \Leftrightarrow x = y \vee x <' y)$

2.3.1 Trichotomy law

Given any numbers $a, b \in C$,it's fulfilled one and only one of the following conditions:

1º $a = b$ 2 º $a <' b$ 3º $b <' a$

3. Squared roots of complex numbers $C - \{0\}$

Given any complex number $a \in C - \{0\}$,we call squared root set of a in the universe $C - \{0\}$,

and we represent it by $\sqrt{a}^{\Delta}$,to the following set of $C - \{0\}$.

$$\sqrt{a}^{\Delta} = \{x \mid x^2 = a\} \quad \text{(If } a = 0,\ \sqrt{a}^{\Delta} = \{0\})$$

The number a ,receives the name of radical and the elements of the set the name of its roots

Given the complex numbers $a = \|a\| \bullet a' = \|a\| \bullet (a'_1 + a'_2 \bullet i)$, and $b = \|b\| \bullet b' = \|b\| \bullet (b'_1 + b'_2 \bullet i)$ belonging to the set $C - \{0\}$,that in canonical form would be:

$$a = \|a\| \bullet i^{c(a)} \bullet a'' = \|a\| \bullet i^{c(a)} \bullet (a''_1 + a''_2 \bullet i)$$
$$b = \|b\| \bullet i^{c(b)} \bullet b'' = \|b\| \bullet i^{c(b)} \bullet (b''_1 + b''_2 \bullet i)$$

3.1 Then the set $\sqrt{a}^{\Delta}$ contains two elements of $C - \{0\}$ that are opposite, and only two.

a)If $1 \leq' a'_1 + a'_2 \bullet i$

$$\sqrt{a}^{\Delta} = \{\sqrt{\|a\|} \bullet (\sqrt{\frac{1+a'_1}{2}} + \sqrt{\frac{1-a'_1}{2}} \bullet i),$$
$$-\sqrt{\|a\|} \bullet (\sqrt{\frac{1+a'_1}{2}} + \sqrt{\frac{1-a'_1}{2}} \bullet i)\}$$

b)If $a'_1 + a'_2 \bullet i <' 1$

$$\sqrt{a}^{\Delta} = \{\sqrt{\|a\|} \bullet (\sqrt{\frac{1+a'_1}{2}} - \sqrt{\frac{1-a'_1}{2}} \bullet i)$$
$$-\sqrt{\|a\|} \bullet (\sqrt{\frac{1+a'_1}{2}} + \sqrt{\frac{1-a'_1}{2}} \bullet i)\}$$

3.1.1. From now on we agree the following:

a)If $1 \leq' a'_1 + a'_2 \bullet i$

$$\sqrt{a} = \sqrt{\|a\|} \bullet (\sqrt{\frac{1+a'_1}{2}} + \sqrt{\frac{1-a'_1}{2}} \bullet i)$$

b)If $a'_1 + a'_2 \bullet i <' \mathbf{1}$

$$\sqrt{a} = \sqrt{\|a\|} \bullet \left(\sqrt{\frac{1+a'_1}{2}} - \sqrt{\frac{1-a'_1}{2}} \bullet i\right)$$

Then in the both cases

$$\sqrt{a}^{\Delta} = \{\sqrt{a}, -\sqrt{a}\}$$

3.2 $\sqrt{-1} = -i$ (By 3.1.1-b)

4.Power of two index roots

Given any complex number $a \in C - \{0\}$. and the set $P = \{2^x \mid x \in N\}$,a unitary operation of

P in C, $f_a : P \to C$ defined, after agreeing, that

$f_a(2^n) = \sqrt[2^n]{a}$,being $2^n \in P$,in the following way:

1º If $n = 0$

$\sqrt[2^n]{a} = a$ is called root of radical a with index in P

2º If we know $\sqrt[2^n]{a} = a$ for $n = m$,then for $n = m+1$

$\sqrt[2^{m+1}]{a} = \sqrt{\sqrt[2^m]{a}}$ if ($a = 0$, $\sqrt[2^n]{a} = 0$)

5**. Power of the complex numbers of $C - \{0\}$ with rational numbers index of denominator power of two**

Let $B = \{\frac{m}{2^n} \mid m \in Z \wedge n \in N\}$ and $a \in C - \{0\}$. ;we call power of base a with index of B a unitary

operation of B in C, $g_a : B \to C$,defined, after agreeing that $g_a(\frac{m}{2^n}) = a^{\frac{m}{2^n}}$,by

$a^{\frac{m}{2^n}} = \sqrt[2^n]{a}^{\,m}$ (if $a = 0$, $a^{\frac{m}{2^n}} = 0$)

And this definition is right, since it's independent of the integer components of the rational number.

For $m, p \in Z$ and $n, h \in N$

$$\frac{m}{2^n} = \frac{p}{2^k} \Rightarrow a^{\frac{m}{2^n}} = a^{\frac{p}{2^k}}$$

6. Adjoining sequences in $k \bullet C'$

The couple of sequences (**a**,**b**) are adjoining in $k \bullet C'$,if and only if, being defined in $k \bullet C'$ $(\forall_{n\in N}(\mathbf{a}_n, \mathbf{b}_n \in k \bullet C'))$

1º The sequence **a** is monotonic increasing and the **b** monotonic decreasing with respect to the modular order

2º $\forall_{n,m\in N}(\mathbf{a}_n \leq' \mathbf{b}_m)$

3º $\mathbf{b} + -\mathbf{a} \in I$ (I subset of infinitesimal sequences)

6.1.If the couple of sequences (**a**,**b**) are adjoining in $k \bullet C'$,both are convergent and **lim a= lim b**.

7. Power of complex sequences with sequences in $B(5)$ as index

Given the complex sequence **a** in wich $\forall_{n\in N}(\mathbf{a}_n \neq 0)$ and the sequence **c** in B ,we call power of base **a** and index **c** and we represent it by $\mathbf{a}^{\mathbf{c}}$ to the complex sequence such that $\forall_{n\in N}((\mathbf{a}^{\mathbf{c}})_n = \mathbf{a}_n^{\mathbf{c}_n})$

7.1. If d is any convergent sequence in B, the sequence $(a^{\mathbf{d}_n})$ is convergent $((a^{\mathbf{d}_n})_n = a^{\mathbf{d}_n})$.

7.2. For any number $a \in C'$ there is a couple of adjoining sequences $(\mathbf{d}, \mathbf{d}')$ in B with its terms in the interval $[-2,2[$ such that $\forall_{n\in N}(i^{\mathbf{d}_n} \leq' a \leq' i^{\mathbf{d}'_n})$

7.3 The couple of sequences $((i^{\mathbf{d}_x}), (i^{\mathbf{d}'_x}))$ is adjoining, and therefore convergent.

8. Power of the complex numbers of $C - \{0\}$ with index in the set R of real numbers

We call power of base $a \in C - \{0\}$ with index in R to an unitary operation of R in C, $f_a : R \to C$, defined, after agreeing, that $f_a(\delta) = a^\delta$, being $\delta \in R$ in the following way:

If d is a sequence in $B(5)$ such that $\lim \mathbf{d} = \delta$

$a^\delta = \lim(a^{\mathbf{d}_x})$ (7.1.) (if $a = 0, a^\delta = 0)$

This definition is right, because it's independent from the sequence that has δ as a limit. Because

$\lim \mathbf{c} = \lim \mathbf{d} \Rightarrow \lim(a^{\mathbf{c}_x}) = \lim(a^{\mathbf{d}_x})$

In the case of being a a real number $(a = a_1 + 0 \bullet i)$,this definition coincides with the one given in the real numbers.

8.1 Given any number $a \in C'$ there exists a real number $\alpha \in [-2, 2[$ and only one such that $i^\alpha = a$.

8.2 Being $a \neq 1$ (and therefore $\alpha \neq 0$);if $\omega = \frac{4}{|\alpha|}$,then $\omega \geq 2$ (if $a \neq i^{-2}, \omega > 2)$

8.2.1 Being $\omega^\bullet = \{x \mid \exists_{t \in Z}(x = \omega \bullet t)\}$ and $a \neq 1$

$a^\gamma = 1 \Leftrightarrow \gamma \in \omega^\bullet$

8.2.2 The operation power of base $a \neq 1$ is periodical, with respect to the index, with period ω (a period)

$\forall_{x \in R}(a^{x+\delta} = a^x) \Leftrightarrow \delta \in \omega^\bullet$

8.2.3 Given the numbers $a, b \in C'$ in which ω is the period of a;if $1 <' a$ there exists a real number $\delta \in [-\frac{\omega}{2}, \frac{\omega}{2}[$ and only one so that

$a^{\delta} = b$

9. Trigonometric function

Being $\omega \in R^{+}$ and taking into account that for $\alpha \in R$

$i^{\frac{4}{\omega}\bullet\alpha} = \mathrm{R}(i^{\frac{4}{\omega}\bullet\alpha}) + \mathrm{I}(i^{\frac{4}{\omega}\bullet\alpha}) \bullet i$

We define the following functions:

a) Sine function in ω (abbreviated in sin_{ω})

$sin_{\omega} : R \to R$ such that

$x \in R \Rightarrow sin_{\omega}x = \mathrm{I}(i^{\frac{4}{\omega}\bullet x}) \qquad (sin_{\omega}(x) = sin_{\omega}x)$

b) Cosine function in ω (abbreviated in $\cos_{\omega}$)

$\cos_{\omega} : R \to R$ such that

$x \in R \Rightarrow \cos_{\omega}x = \mathrm{R}(i^{\frac{4}{\omega}\bullet x})$

c) Tangent function in ω (abbreviated in $\tan_{\omega}$)

Being $(\frac{\omega}{4})^{*} = \{x \mid \exists_{t \in Z}(x = \frac{\omega}{4} \bullet (2 \bullet t + 1))\}$

$\tan\omega : R - (\frac{\omega}{4})^{*} \to R$ such that

$x \in R - (\frac{\omega}{4})^{*} \Rightarrow \tan_{\omega}x = \frac{\sin_{\omega}x}{\cos_{\omega}x}$

d) Cotangent function in ω (abbreviated in $\cot_{\omega}$)

Being $\left(\frac{\omega}{2}\right)^{\bullet} = \{x \mid \exists_{t \in Z}(x = \frac{\omega}{2} \bullet t)\}$

$\cot_{\omega} : R - \left(\frac{\omega}{2}\right)^{\bullet} \to R$ such that

$x \in R - \left(\frac{\omega}{2}\right)^{\bullet} \Rightarrow \cot_{\omega} x = \frac{\cos_{\omega} x}{\sin_{\omega} x}$

The name of argument of the correspondent function is given to each element of R, $R - \left(\frac{\omega}{4}\right)^{*}$ or $R - \left(\frac{\omega}{2}\right)^{\bullet}$

9.1. Trigonometric functions of arguments addition

a) $\sin_{\omega}(\alpha + \beta) = \sin_{\omega}\alpha \bullet \cos_{\omega}\beta + \cos_{\omega}\alpha \bullet \sin_{\omega}\beta$

b) $\cos_{\omega}(\alpha + \beta) = \cos_{\omega}\alpha \bullet \cos_{\omega}\beta - \sin_{\omega}\alpha \bullet \sin_{\omega}\beta$

Proof:

$\cos_{\omega}(\alpha + \beta) + \sin_{\omega}(\alpha + \beta) \bullet i = i^{\frac{4}{\omega} \bullet (\alpha + \beta)} = i^{\frac{4}{\omega} \bullet \alpha + \frac{4}{\omega} \bullet \beta} = i^{\frac{4}{\omega} \bullet \alpha} \bullet i^{\frac{4}{\omega} \bullet \beta} = (\cos_{\omega}\alpha + \sin_{\omega}\alpha \bullet i) \bullet (\cos_{\omega}\beta + \sin_{\omega}\beta \bullet i) = (\cos_{\omega}\alpha \bullet \cos_{\omega}\beta - \sin_{\omega}\alpha \bullet \sin_{\omega}\beta) + (\sin_{\omega}\alpha \bullet \cos_{\omega}\beta + \cos_{\omega}\alpha \bullet \sin_{\omega}\beta) \bullet i$

10. Being $\omega, \omega' \in R^{+}$

a) $\sin_{\omega}(\omega \bullet \alpha) = \sin_{\omega'}(\omega' \bullet \alpha)$

b) $\cos_{\omega}\omega \bullet \alpha) = \cos_{\omega'}(\omega' \bullet \alpha)$

10.1 a) $\sin_{\omega}\alpha = \sin_{\omega'}\left(\frac{\omega'}{\omega} \bullet \alpha\right)$

b) $\cos_{\omega}\alpha = \cos_{\omega'}\left(\frac{\omega'}{\omega} \bullet \alpha\right)$

11. The following couple of real sequences are adjoining, and therefore convergent,

$$\left(\left(\frac{2^{n+3}}{\omega}\bullet\sin_{\omega}\frac{\omega}{2^{n+3}}\right),\left(\frac{2^{n+3}}{\omega}\bullet\tan_{\omega}\frac{\omega}{2^{n+3}}\right)\right)$$

12. The mapping $\lambda : R^{+} \to R^{+}$ is bijective

so that $\omega \in R^{+} \Rightarrow \lambda(\omega) = \lim\left(\frac{2^{n+3}}{\omega}\bullet\sin_{\omega}\frac{\omega}{2^{n+3}}\right)$

12.1 For every $\omega, \omega' \in R^{+}$ and agreeing that $\lambda(\omega) = \lambda_{\omega}$

$\omega \bullet \lambda_{\omega} = \omega' \bullet \lambda_{\omega'}$ (9 and 10-a)

12.2 Starting from the equality $\omega \bullet \lambda_{\omega} = 2 \bullet \lambda_{2}$, we can solve the following interesting problem:

For which number ω is accomplished that $\lambda_{\omega} = 1$?

The solution is obvious, for $\omega = 2 \bullet \lambda_{2}$

This problem, with geometrical language, was already posed by the ancient Egyptians, and they knew that

$$3{,}1 < \lambda_{2} < 3{,}2$$

Nowadays to this important real number is named (π).

12.3 Boundary mark of the number λ_{2} (π)

We know that

$$2^{n+2}\bullet\sin_{2}\left(\frac{1}{2^{n+2}}\right) < \lambda_{2} < 2^{n+2}\bullet\tan_{2}\left(\frac{1}{2^{n+2}}\right)$$

Then, giving successive values to the natural number n $, from zero, we will obtain two sequences of numbers that converge to λ_{2} (π)

12.4 To facilitate the calculation of λ_{2} (π) , we take into account the following equalities:

a) $2^{n+2} \bullet sin_2(\frac{1}{2^{n+2}}) = 2 : (\cos_2(\frac{1}{2^2}) \bullet \cos_2(\frac{1}{2^3}) \bullet\bullet \bullet \cos_2(\frac{1}{2^{n+2}}))$

b) $2^{n+2} \bullet \tan_2(\frac{1}{2^{n+2}}) = 2:(\cos_2(\frac{1}{2^2}) \bullet \cos_2(\frac{1}{2^3}) \bullet\bullet \bullet \cos_2(\frac{1}{2^{n+2}}) \bullet \cos_2(\frac{1}{2^{n+2}}))$

c)For $n = 0$, $\cos_2(\frac{1}{2^2}) = \frac{\sqrt{2}}{2}$

d)If we know $\cos_2(\frac{1}{2^{n+2}})$ for $n = m$,the for $n = m+1$

$$\cos_2 \frac{1}{2^{m+3}} = \sqrt{(1 + \cos_2 \frac{1}{2^{m+2}}) : 2}$$

12.5 Having into account the previous recursive calculation, we obtain from a pocket computer for $n = 16$ that

$$3,141592653 < \lambda_2 < 3,141592654$$

13.We agree that

a) $\sin_{2\bullet\pi}\alpha = \sin\alpha$ b) $\cos_{2\bullet\pi}\alpha = \cos\alpha$ c) $\tan_{2\bullet\pi}\alpha = \tan\alpha$ d) $\cot_{2\bullet\pi}\alpha = \cot\alpha$

14 By definition

$$e^{\alpha\bullet i} = i^{\frac{4}{2\bullet\pi}\bullet\alpha} \quad (e^{a_1+a_2\bullet i} = e^{a_1} \bullet i^{\frac{4}{2\bullet\pi}\bullet a_2})$$

15.All the properties here exposed and the ones omitted are proved in the book "Didáctica del número complejo" published by "Servicio de Publicaciones de la Universidad de Oviedo".
I.S.B.N: 84-7468-835-3. D.L: AS/3751-94 \ (240 pp.)